Goldstein 2.4

~~Goldstein 2.14~~

Parameterize path on spherical surface by

$$\theta = \theta(t), \quad \phi = \phi(t)$$

Then the action is given by

$$I = \int_{t_0}^{t_1} \sqrt{\dot{\theta}^2 + \dot{\phi}^2}\, dt$$

This identifies Lagrangian $L = \sqrt{\dot{\theta}^2 + \dot{\phi}^2}$, it gives equation of motion

$$\frac{d}{dt}\left[\frac{\dot{\theta}}{\sqrt{\dot{\theta}^2 + \dot{\phi}^2}}\right] = 0, \quad \frac{d}{dt}\left[\frac{\dot{\phi}}{\sqrt{\dot{\theta}^2 + \dot{\phi}^2}}\right] = 0.$$

which suggests introduction of constants α, β, such that

$$\frac{\dot{\theta}}{\sqrt{\dot{\theta}^2 + \dot{\phi}^2}} = \alpha, \quad \frac{\dot{\phi}}{\sqrt{\dot{\theta}^2 + \dot{\phi}^2}} = \beta$$

$$\Rightarrow \frac{\dot{\theta}^2}{\dot{\theta}^2 + \dot{\phi}^2} = \alpha^2, \quad \frac{1}{1 + \left(\frac{\dot{\theta}}{\dot{\phi}}\right)^2} = \alpha^2,$$

that is, $\frac{\dot{\theta}}{\dot{\phi}}$ is a constant, thus $\frac{\theta}{\phi}$ is constant (if WLOG, they both start at 0: $\theta(t_0) = 0$, $\phi(t_0) = 0$).

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